

The Statistics of Sampling

By

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The Statistics of Sampling

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FOREWORD

In the autumn of 1923 while a Fellow at the University of Michigan Professor Harry C. Carver proposed to me the problem of characterizing the distributions of the first, the second, the third, and the fourth moments. The most pressing demand was that of characterizing the distribution of the first and second moments of samples drawn from a finite population, and by a method that was based upon rather elementary mathematics.

This paper is an abridgment of my solution to the problem. If the paper has any merit, much of the credit for it is due Professor Carver without whose aid the investigation could not have been extended. Of course if there are any errors, I alone am responsible.

C. H. Richardson



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Chapter I

INTRODUCTION

1. Introductory Remarks. The Problem.

Toward the numerical description of any statistical distribution, we must recognize two points of view. We may view the description of the given distribution as an end in itself, or we may regard it as a sample drawn from and representative of a larger group that we shall call the parent population, or the parent universe. Generally, the latter view obtains, and we desire to form a judgment of the parent universe from a study of a sample from it.

Let us assume that it is either impracticable or impossible to measure all the members of the parent universe for a given characteristic. We are compelled to choose a sample. Assume the parent population has s members from which samples of r , $r < s$, are taken. We may obtain sCr samples. Each of these samples has a mean or first moment. The sCr samples give us a distribution of sCr sample first moments. Similarly, each sample has a second moment, and the sCr samples give us a distribution of sCr sample second moments. Proceeding in this way we obtain distributions of sample third moments, sample fourth moments, et cetera.

The problem of sampling has been called by Karl Pearson the fundamental problem in statistics. Often our only statistical knowledge of the parent universe is obtained from a study of samples drawn from it. Stated in rather general terms, the question is: how well does the sample describe the parent universe? More precisely, our problem is: how do the moments computed from the sample compare with the moments computed from the parent population?

When s is finite, we say that we are sampling from a limited supply or from a finite parent

universe; when s is assumed infinite, we say that we are sampling from an unlimited supply or from an infinite parent universe.

Due to the fact that the algebraical simplifications are well nigh impossible, especially for the distributions of higher moments, when sampling from a limited supply, the papers that have been written have been concerned in the main with the case in which s is infinite.

Papers by "Student", Tschouproff, Thiele, and others who have investigated the problem have assumed that the samples were drawn from an infinite universe.

We shall show how one may secure any moment of the distribution of first moments when s is finite, then by permitting s to become infinite we shall obtain the corresponding moments when sampling from an unlimited supply. Further, we shall obtain the earlier moments of the distribution of second moments for the case in which s is finite. For the higher moments, we shall restrict the problem to the case in which s is infinite.

2. Notation

$X_1, i = 1, 2, \dots, s$, denotes the s variates measured from a fixed point as origin.

$M_x = \mu_1'$ the mean of the parent population

$x_1, i = 1, 2, \dots, s$, denotes the s variates measured from the mean of the parent population,

$$\text{i.e., } x_i = X_i - M_x$$

μ_n' = the n th moment of the parent population about a fixed point.

μ_n = the n th moment of the parent population about M_x .

$\sigma = \sqrt{\mu_2}$ = the standard deviation of the parent population.

$\alpha_n = \frac{\mu_n}{\sigma^n}$ = the n th moment of the parent population about M_x expressed in standard units.

$\mu'_{m,n}$ = the mth moment of the distribution of nth moments about a fixed point.

$\mu_{m,n}$ = the mth moment of the distribution of nth moments about its mean.

σ_{μ_n} = the standard deviation of the distribution of nth moments.

$m'_{n,k}$ = the nth moment of the kth sample about a fixed point.

$m_{n,k}$ = the nth moment of the kth sample about the mean of the sample.

$$p_i = \frac{r(i)}{s(i)} = \frac{r(r-1)(r-2)\cdots(r-i+1)}{s(s-1)(s-2)\cdots(s-i+1)}$$

Chapter II

THE DISTRIBUTION OF MEANS

3. Moments When s and r are Finite

Consider the parent population $X_1, X_2, \dots, X_s$, and let $m'_{1,k}$ be the k th sample mean. Then¹

$$m'_{1,1} = \frac{1}{r} [X_1 + X_2 + \dots + X_{r-1} + X_r] = \frac{1}{r} \sum_{i=1}^r X_i$$

$$m'_{1,2} = \frac{1}{r} [X_1 + X_2 + \dots + X_{r-1} + X_{r+1}] = \frac{1}{r} \sum_{i=1}^{r+1} X_i$$

.....

$$m'_{1,sCr} = \frac{1}{r} [X_{s-r+1} + X_{s-r+2} + \dots + X_s] = \frac{1}{r} \sum_{i=1}^{r:sCr} X_i$$

The arithmetic mean of these sample means is given by

$$(1) \mu'_{1,1} = \frac{1}{sCr} \sum_{k=1}^{sCr} m'_{1,k} = \frac{1}{sCr} \cdot \frac{1}{r} \frac{rCl \cdot sCr}{sCl} \sum_{i=1}^s X_i = \frac{\sum_{i=1}^s X_i}{s} = M_x$$

That is, the mean of the sCr sample means is equal to the mean of the parent population.

We now proceed to find other statistical characteristics of the distribution of means. To accomplish this, we transform our variates to the mean of the parent population (which equals the mean of the distribution of means) as origin. That is, we have

$$x_i = X_i - M_x, \quad i = 1, 2, 3, \dots, s,$$

1. We denote by $\sum_{i=1}^s X_i$ the sum of the s variates of the parent population, and by $\sum_{i=1}^{r:i} X_i$ the sum of the r variates in the i th sample.

and

$$\sum_{i=1}^s x_i = \sum x = 0.$$

Let

$$m_{1,k} = \frac{1}{r} \sum x^{r;k}$$

denote the k th sample mean of r variates chosen from the transformed parent population $x_1, x_2, \dots, x_s$. Then,

$$(m_{1,1})^2 = \frac{1}{r^2} \left[\sum x^{r;1} x^2 + 2 \sum x_1 x_j^{r;1} \right]$$

$$(m_{1,2})^2 = \frac{1}{r^2} \left[\sum x^{r;2} x^2 + 2 \sum x_1 x_j^{r;2} \right]$$

$$\dots\dots\dots$$

$$(m_{1,sCr})^2 = \frac{1}{r^2} \left[\sum x^{r;sCr} x^2 + 2 \sum x_1 x_j^{r;sCr} \right]$$

Therefore

$$\mu_{2,1} = \frac{1}{sCr} \sum_{k=1}^{sCr} (m_{1,k})^2 = \frac{1}{sCr} \cdot \frac{1}{r^2} \left[\frac{rC_1 \cdot sCr}{sC_1} \sum x^2 + 2 \frac{rC_2 \cdot sCr}{sC_2} \sum x_1 x_j \right]$$

$$r^2 \mu_{2,1} = \left[\frac{r}{s} \sum x^2 + 2 \frac{r(2)}{s} \sum x_1 x_j \right] = \left[\rho_1 \sum x^2 + 2 \rho_2 \sum x_1 x_j \right]$$

Using the second order summations

$$\sum x^2 = s\mu_2$$

$$(\sum x)^2 = 0 = \sum x^2 + 2 \sum x_1 x_j = s\mu_2 + 2 \sum x_1 x_j,$$

$$2 \sum x_1 x_j = -s\mu_2,$$

we have immediately that

$$(2) \quad r^2 \mu_{2,1} = s\mu_2 [\rho_1 - \rho_2]$$

from which we find

$$\sigma_{\mu_1} = \sigma \sqrt{\frac{s-r}{r(s-1)}}$$

Similarly we have

$$\mu_{3,1} = \frac{1}{sCr} \sum_{k=1}^{sCr} (m_{1,k})^3 = \frac{1}{sCr} \cdot \sum_{k=1}^{sCr} \frac{1}{r^3} \left[\sum_{i,k}^{r,k} x^3 + 3 \sum_{i,k}^{r,k} x_1^2 x_j + 6 \sum_{i,k}^{r,k} x_1 x_j x_k \right]$$

$$r^3 \mu_{3,1} = [\rho_1 \sum x^3 + 3\rho_2 \sum x_1^2 x_j + 6\rho_3 \sum x_1 x_j x_k]$$

Using the third order summations

$$\sum x^3 = s\mu_3$$

$$\sum x^2 \sum x = 0 = \sum x^3 + \sum x_1^2 x_j = s\mu_3 + \sum x_1^2 x_j$$

$$\sum x_1 x_j \sum x = 0 = \sum x_1^2 x_j + 3 \sum x_1 x_j x_k,$$

that is,

$$\sum x^3 = s\mu_3$$

$$\sum x_1^2 x_j = -s\mu_3$$

$$3 \sum x_1 x_j x_k = s\mu_3$$

we find

$$(3) \quad r^3 \mu_{3,1} = s\mu_3 [\rho_1 - 3\rho_2 + 2\rho_3]$$

By the same method we have

$$\begin{aligned} \mu_{4,1} &= \frac{1}{sCr} \sum_{k=1}^{sCr} (m_{1,k})^4 = \frac{1}{sCr} \sum_{k=1}^{sCr} \frac{1}{r^4} \left[\sum_{i,k}^{r,k} x^4 + 4 \sum_{i,k}^{r,k} x_1^3 x_j \right. \\ &\quad \left. + 6 \sum_{i,k}^{r,k} x_1^2 x_j^2 + 12 \sum_{i,k}^{r,k} x_1^2 x_j x_k + 24 \sum_{i,k}^{r,k} x_1 x_j x_k x_l \right] \end{aligned}$$

$$\begin{aligned} r^4 \mu_{4,1} &= [\rho_1 \sum x^4 + 4\rho_2 \sum x_1^3 x_j + 6\rho_2 \sum x_1^2 x_j^2 + 12\rho_3 \sum x_1^2 x_j x_k \\ &\quad + 24\rho_4 \sum x_1 x_j x_k x_l] \end{aligned}$$

Using the fourth order summations

$$\sum x^4 = s\mu_4$$

$$\sum x^3 \cdot \sum x = 0 = \sum x^4 + \sum x_1^3 x_j = s\mu_4 + \sum x_1^3 x_j$$

$$(\sum x^2)^2 = (s\mu_2)^2 = \sum x^4 + 2 \sum x_1^2 x_j^2 = s\mu_4 + 2 \sum x_1^2 x_j^2$$

$$\begin{aligned} \sum x^2 (\sum x)^2 &= 0 = \sum x^2 [\sum x^2 + 2 \sum x_1 x_j] \\ &= (\sum x^2)^2 + 2 \sum x_1^3 x_j + 2 \sum x_1^2 x_j x_k \end{aligned}$$

$$(\sum x)^4 = 0 = \sum x^4 + 4 \sum x_1^3 x_j + 6 \sum x_1^2 x_j^2 + 12 \sum x_1^2 x_j x_k + 24 \sum x_1 x_j x_k x_l$$

that is, using

$$\sum x^4 = s\mu_4 \quad \sum x_1^3 x_j = -s\mu_4$$

$$2 \sum x_1^2 x_j^2 = s^2 \mu_2^2 - s\mu_4 \quad 2 \sum x_1^2 x_j x_k = 2s\mu_4 - s^2 \mu_2^2$$

$$24 \sum x_1 x_j x_k x_l = -6s\mu_4 + 3s^2 \mu_2^2$$

we obtain

$$r^4 \mu_{4,1} = s\mu_4 [\rho_1 - 7\rho_2 + 12\rho_3 - 6\rho_4] + 3s^2 \mu_2^2 [\rho_2 - 2\rho_3 + \rho_4]$$

For many purposes the above notation is satisfactory, especially for the distribution of means, but when we encounter the moments of the distributions of second, third, and fourth moments we are compelled to seek a simplified notation.¹

Continuing, we shall still consider the variates of our parent population to be referred to its mean M_x as origin, that is, so that

$$\sum_{i=1}^s x_i = (1) = 0.$$

[Note that $(1) = 0$ is true only when the parent population is under consideration.]

To acquaint the reader with the simplicity in applying the partition notation,² we shall

1. The reader should turn to Appendix A and master the method explained there before proceeding further in this text.
2. The subscript of ρ is the number of elements in the term. Thus $\sum x_i^2 x_j x_k = (2.1^2)$ has three elements.

compute $\mu_{4,1}$ again. Recalling that

$$m_{1,k} = \frac{1}{r} \sum_{i=1}^r x_i^k$$

denotes the k th sample mean of r variates chosen from the parent population $x_1, x_2, \dots, x_s$, we have

$$(m_{1,k})^4 = \frac{1}{r^4} [(4) + 4(3.1) + 6(2^2) + 12(2.1^2) + 24(1^4)]$$

from which it follows that

$$\begin{aligned} \mu_{4,1} = \frac{1}{sCr} \sum_{k=1}^{sCr} (m_{1,k})^4 &= \frac{1}{r^4} [\rho_1(4) + 4\rho_2(3.1) + 6\rho_2(2^2) \\ &+ 12\rho_3(2.1^2) + 24\rho_4(1^4)] \end{aligned}$$

Applying the proper fourth order products found in Appendix B, we have

$$(4) \quad r^4 \mu_{4,1} = s\mu_4 [\rho_1 - 7\rho_2 + 12\rho_3 - 6\rho_4] + 3s^2 \mu_2^2 [\rho_2 - 2\rho_3 + \rho_4]$$

Similarly we find by the multinomial theorem

$$\begin{aligned} r^5 (m_{1,k})^5 &= (5) + 5(4.1) + 10(3.2) + 20(3.1^2) + 30(2^2.1) \\ &+ 60(2.1^3) + 120(1^5) \end{aligned}$$

from which

$$\begin{aligned} r^5 \mu_{5,1} &= \rho_1(5) + 5\rho_2(4.1) + 10\rho_2(3.2) + 20\rho_3(3.1^2) \\ &+ 30\rho_3(2^2.1) + 60\rho_4(2.1^3) + 120\rho_5(1^5) \end{aligned}$$

Applying the proper fifth order products we have

$$\begin{aligned} (5) \quad r^5 \mu_{5,1} &= s\mu_5 [\rho_1 - 15\rho_2 + 50\rho_3 - 60\rho_4 + 24\rho_5] \\ &+ 10s^2 \mu_2 \mu_3 [\rho_2 - 4\rho_3 + 5\rho_4 - 2\rho_5] \end{aligned}$$

Proceeding in the same manner we have

$$\begin{aligned}
 r^6(m_1, k)^6 &= (6) + 6(5.1) + 15(4.2) + 30(4.1^2) + 20(3^2) \\
 &\quad + 60(3.2.1) + 120(3.1^3) + 90(2^3) + 180(2^2.1^2) \\
 &\quad + 360(2.1^4) + 720(1^6).
 \end{aligned}$$

from which

$$\begin{aligned}
 (6) \quad r^6 \mu_{6,1} &= s \mu_6 [\rho_1 - 31\rho_2 + 180\rho_3 - 390\rho_4 + 360\rho_5 - 120\rho_6] \\
 &\quad + 15s^2 \mu_4 \mu_2 [\rho_2 - 8\rho_3 + 19\rho_4 - 18\rho_5 + 6\rho_6] \\
 &\quad + 10s^2 \mu_3^2 [\rho_2 - 6\rho_3 + 13\rho_4 - 12\rho_5 + 4\rho_6] \\
 &\quad + 15s^3 \mu_2^3 [\rho_3 - 3\rho_4 + 3\rho_5 - \rho_6]
 \end{aligned}$$

The same method yields

$$\begin{aligned}
 (7) \quad r^7 \mu_{7,1} &= s \mu_7 [\rho_1 - 63\rho_2 + 602\rho_3 - 2100\rho_4 + 3360\rho_5 \\
 &\quad - 2520\rho_6 + 720\rho_7] \\
 &\quad + 21s^2 \mu_5 \mu_2 [\rho_2 - 16\rho_3 + 65\rho_4 - 110\rho_5 + 84\rho_6 - 24\rho_7] \\
 &\quad + 35s^2 \mu_4 \mu_3 [\rho_2 - 10\rho_3 + 35\rho_4 - 56\rho_5 + 42\rho_6 - 12\rho_7] \\
 &\quad + 105s^3 \mu_3 \mu_2^2 [\rho_3 - 5\rho_4 + 9\rho_5 - 7\rho_6 + 2\rho_7] \\
 (8) \quad r^8 \mu_{8,1} &= s \mu_8 [\rho_1 - 127\rho_2 + 1932\rho_3 - 10206\rho_4 + 25200\rho_5 \\
 &\quad - 31920\rho_6 + 20160\rho_7 - 5040\rho_8] \\
 &\quad + 28s^2 \mu_6 \mu_2 [\rho_2 - 32\rho_3 + 211\rho_4 - 570\rho_5 + 750\rho_6 \\
 &\quad - 480\rho_7 + 120\rho_8] \\
 &\quad + 56s^2 \mu_5 \mu_3 [\rho_2 - 18\rho_3 + 97\rho_4 - 240\rho_5 + 304\rho_6 \\
 &\quad - 192\rho_7 + 48\rho_8] \\
 &\quad + 35s^2 \mu_4^2 [\rho_2 - 14\rho_3 + 73\rho_4 - 180\rho_5 + 228\rho_6 - 144\rho_7 + 36\rho_8] \\
 &\quad + 210s^3 \mu_4 \mu_2^2 [\rho_3 - 9\rho_4 + 27\rho_5 - 37\rho_6 + 24\rho_7 - 6\rho_8]
 \end{aligned}$$

$$\begin{aligned}
 &+ 280s^3\mu_3^2\mu_2[\rho_3-7\rho_4+19\rho_5-25\rho_6+16\rho_7-4\rho_8] \\
 &+ 105s^4\mu_2^4[\rho_4-4\rho_5+6\rho_6-4\rho_7+\rho_8]
 \end{aligned}$$

4. Moments When s is Infinite and r Finite

If, in the preceding equations (1), (2), ---, (8) we let s increase without limit we find

$$(9) \left\{ \begin{aligned}
 \mu_{1,1}' &= \mu_1' \\
 r^2\mu_{2,1} &= r\mu_2 \text{ and } \sigma\mu_1 = \frac{\sigma}{\sqrt{r}} \\
 r^3\mu_{3,1} &= r\mu_3 \\
 r^4\mu_{4,1} &= r\mu_4 + 3r^{(2)}\mu_2^2 \\
 r^5\mu_{5,1} &= r\mu_5 + 10r^{(2)}\mu_3\mu_2 \\
 r^6\mu_{6,1} &= r\mu_6 + r^{(2)}[15\mu_4\mu_2 + 10\mu_3^2] + 15r^{(3)}\mu_2^3 \\
 r^7\mu_{7,1} &= r\mu_7 + r^{(2)}[21\mu_5\mu_2 + 35\mu_4\mu_3] + 105r^{(3)}\mu_3\mu_2^2 \\
 r^8\mu_{8,1} &= r\mu_8 + r^{(2)}[28\mu_6\mu_2 + 56\mu_5\mu_3 + 35\mu_4^2] \\
 &\quad + r^{(3)}[210\mu_4\mu_2^2 + 28\mu_3^2\mu_2] + 105r^{(4)}\mu_2^4
 \end{aligned} \right.$$

Equations (9), which give the moments of the distribution of means in terms of the moments of the unlimited parent universe, were obtained from similar relations in which the parent universe was limited. This method was found to be extremely tedious.

For the case in which s is unlimited, we can avoid the tedium by employing a theorem proved in Appendix A which, on account of its power, I have called the Master Theorem. Applying the Master Theorem we find it entirely unnecessary to take cognizance of any term that involves unity in the partition. Thus we can immediately write down $\mu_{9,1}$:

$$(10) \quad r^9\mu_{9,1} = \frac{9!}{9!} r^{(1)}\mu_9 + \frac{9!}{7!2!} r^{(2)}\mu_7\mu_2 + \frac{9!}{6!3!} r^{(2)}\mu_6\mu_3$$

$$\begin{aligned}
& + \frac{9!}{5!4!} r^{(2)} \mu_5 \mu_4 + \frac{9!}{5!(2!)^2} r^{(3)} \mu_5 \mu_2^2 \\
& + \frac{9!}{4!3!2!} r^{(3)} \mu_4 \mu_3 \mu_2 + \frac{9!}{(3!)^3} r^{(3)} \mu_3^3 \\
& + \frac{9!}{3!(2!)^3} r^{(4)} \mu_3 \mu_2^3
\end{aligned}$$

which reduces to

$$\begin{aligned}
(11) \quad r^9 \mu_{9,1} &= r \mu_9 + r^{(2)} [36 \mu_7 \mu_2 + 84 \mu_6 \mu_3 + 126 \mu_5 \mu_4] \\
&+ r^{(3)} [378 \mu_5 \mu_2^2 + 1260 \mu_4 \mu_3 \mu_2 + 280 \mu_3^3] \\
&+ r^{(4)} 1260 \mu_3 \mu_2^3
\end{aligned}$$

In terms of standard moments we have

$$(12) \left\{ \begin{aligned}
\alpha_{2,1} &= 1 \\
r^{1/2} \alpha_{3,1} &= \alpha_3 \\
r^2 \alpha_{4,1} &= r \alpha_4 + 3r^{(2)} \\
r^{5/2} \alpha_{5,1} &= r \alpha_5 + 10r^{(2)} \alpha_3 \\
r^3 \alpha_{6,1} &= r \alpha_6 + r^{(2)} [15\alpha_4 + 10\alpha_3^2] + 15r^{(3)} \\
r^{7/2} \alpha_{7,1} &= r \alpha_7 + r^{(2)} [21\alpha_5 + 35\alpha_4 \alpha_3] + 105r^{(3)} \alpha_3 \\
r^4 \alpha_{8,1} &= r \alpha_8 + r^{(2)} [28\alpha_6 + 56\alpha_5 \alpha_3 + 35\alpha_4^2] \\
&+ r^{(3)} [210\alpha_4 + 28\alpha_3^2] + 105r^{(4)}
\end{aligned} \right.$$

It is interesting to note that if r is so large that $1/\sqrt{r}$, $1/r$, etc., may be neglected equations (12) become

$$(13) \left\{ \begin{array}{l} \alpha_{2,1} = 1 \\ \alpha_{3,1} = 0 \\ \alpha_{4,1} = 3 \\ \alpha_{5,1} = 0 \\ \alpha_{6,1} = 1.3.5 \\ \alpha_{7,1} = 0 \\ \alpha_{8,1} = 1.3.5.7 \end{array} \right.$$

which are the moments of the normal curve.¹ That is to say, if large samples are taken from an infinite parent universe, we may expect the distribution of means of the samples to be closely normal.

It has been shown² that for the Pearson Type III Curve the relation

$$\alpha_{n+1} = n[\alpha_{n-1} + \frac{\alpha_3}{2} \alpha_n]$$

obtains among its moments. Employing this relation, writing $\alpha_3 = 2\gamma$, we obtain from (12)

$$(14) \left\{ \begin{array}{l} \alpha_{2,1} = 1 \\ r^{1/2} \alpha_{3,1} = \alpha_3 \\ r \alpha_{4,1} = 3[r + \gamma] \\ r^{3/2} \alpha_{5,1} = 2\alpha_3[5r + 3\gamma] \\ r^2 \alpha_{6,1} = 5[3r^2 + 13r\gamma + 6\gamma^2] \\ r^{5/2} \alpha_{7,1} = 3\alpha_3[35r^2 + 77r\gamma + 30\gamma^2] \\ r^3 \alpha_{8,1} = 7[15r^3 + 2\gamma(49r^2 + 108r - 72) + 261r\gamma^2 + 90\gamma^3] \end{array} \right.$$

1. Handbook of Mathematical Statistics, H. L. Rietz (editor), p. 97.

2. Rietz, H. L., op. cit., page 106.

as the expected moments of the distribution of means when samples of r are chosen from an unlimited parent universe of Pearson Type III.

Now equations (14) satisfy the condition

$$\alpha_{n+1,1} = n[\alpha_{n-1,1} + \frac{\alpha_{3,1}}{2}\alpha_{n,1}]$$

among the moments that a distribution belongs to Pearson Type III. Therefore, if samples are taken from a Pearson Type III parent population, the distribution of the means of those samples is also a Pearson Type III.

5. Numerical Example

It has been shown by Charlier, Thiele, Carver and others that a frequency distribution may be represented by

$$(15) \quad F(t) = \frac{N}{\sigma} [\varphi(t) - \frac{c_3}{3!} \varphi^{(3)}(t) + \frac{c_4}{4!} \varphi^{(4)}(t) - \dots]$$

where $\varphi(t)$ is the probability function

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

and $c_3 = \alpha_3$, $c_4 = \alpha_4 - 3$.

In order to compare the theory developed in this chapter with experiment, we have chosen for our parent universe the well-known skew distribution of the *Ranunculus Bulbosus* of DeVries.

Number of Petals X	Frequency $f(x)$
5	133
6	55
7	23
8	7
9	2
10	2
Total	222

For this distribution we have computed the constants

$$M_x = \mu_1' = 5.6306 \ 3060$$

$$\sigma = 0.9579 \ 2501$$

$$\alpha_3 = 1.8700 \ 4791$$

$$\alpha_4 = 7.0094 \ 3132$$

From this parent population in which $s = 222$ we have drawn one thousand samples with $r = 37$. For each sample we secured values of $\Sigma X, \Sigma X^2, \Sigma X^3, \Sigma X^4$. We propose here to predict a distribution of ΣX using equations (1), (2), (3), and (4) to determine the constants in our prediction-equation (15), then to compare the result with the actual distribution of ΣX which we obtained by random drawing.

Let

z = a sample mean

L = the class lower limit of a sample ΣX in actual units

$$t = \frac{\frac{L}{37} - M_z}{\sigma_z} = \frac{L - 37M_z}{37\sigma_z} = \text{the class lower limit of a sample } \Sigma X \text{ in standard units}$$

From equations (1), (2), (3), and (4) we obtained

$$M_z = 5.6306 \ 3060$$

$$\sigma_z = 0.1440 \ 8553$$

$$\alpha_{3,z} = 0.2260 \ 4754$$

$$\alpha_{4,z} = 2.9959 \ 1208$$

and the equation of the distribution of means becomes

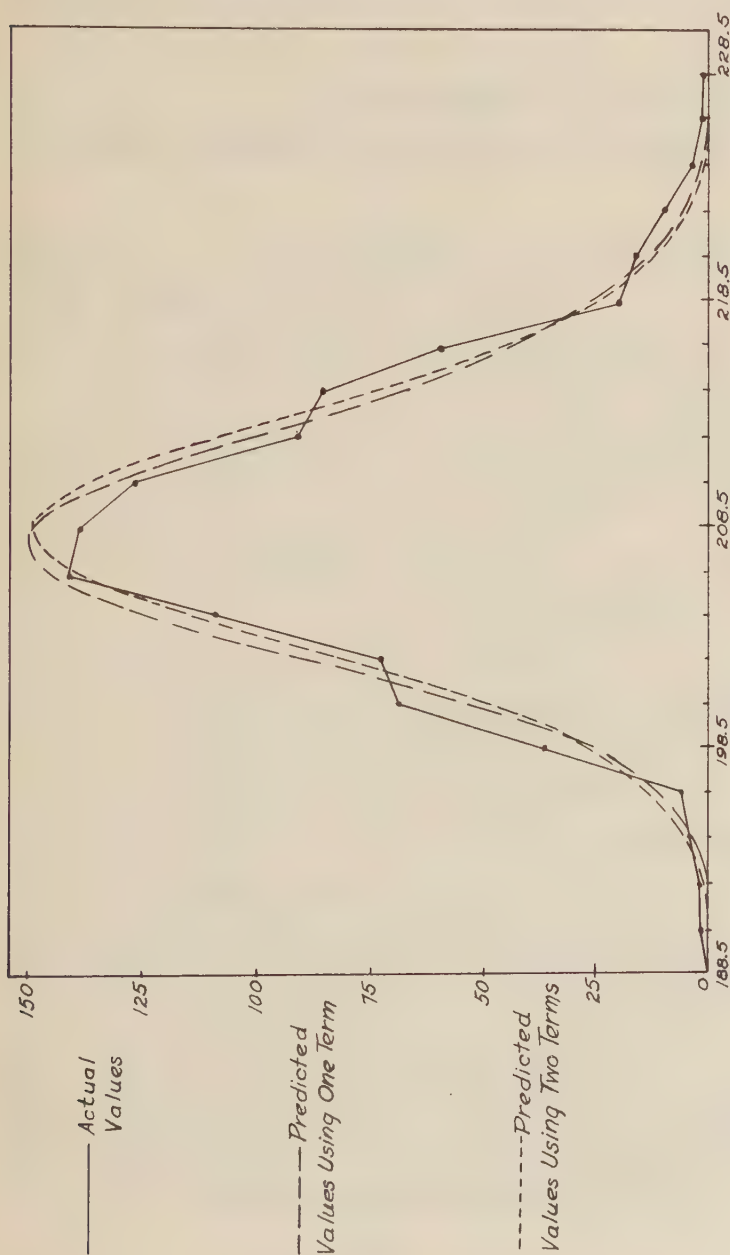
$$F(t) = \frac{1000}{z} [\varphi(t) - 0.0376746\varphi^{(3)}(t) - 0.00017033\varphi^{(4)}(t)]$$

We obtain in the following table three predicted distributions and show in the last column the actual distribution that resulted in the drawing of one thousand samples.

The graphical representation exhibits the theoretical and the actual distributions.

Boundary Point of Class Interval		Integral of Generating Function	Second Derivative	Third Derivative	Integral of Expansion of		Predicted Unit Frequency			Predicted Frequency of X			Act'l. Freq. of X
L	t	$\int_{-\infty}^t \phi(t) dt$	$\phi^{(2)}(t)$	$\phi^{(3)}(t)$	Two Terms $\frac{C_2}{3!}(4)$	Three Terms $\frac{C_3}{3!}(4) + \frac{C_4}{4!}(5)$	One Term $\Delta(3)$	Two Terms $\Delta(6)$	Three Terms $\Delta(7)$	One Term 1000 (8)	Two Terms 1000 (9)	Three Terms 1000 (10)	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
187.5	-3.9078	.0000465	.0027502	.0092409	-.00006	-.00006	.00016	-.00007	-.00008	0	0	0	0
189.5	-3.5327	.0002057	.0089308	.0260527	-.00013	-.00014	.00059	+.00000	+.00000	1	0	0	1
191.5	-3.1575	.0007957	.0244792	.0600574	-.00013	-.00014	.00190	.00072	.00070	2	1	1	1
193.5	-2.7824	.0026381	.0560522	.1096898	+.00059	-.00056	.00534	.00347	.00348	5	3	3	5
195.5	-2.4072	.0080382	.1055313	.1480626	.00406	+.00404	.01303	.01105	.01105	13	11	11	6
197.5	-2.0321	.0210726	.1583710	+.1161424	.01511	.01509	.02770	.02701	.02704	28	27	27	37
199.5	-1.6569	.0487716	.1764527	-.0426723	.04212	.04213	.05119	.05359	.05363	51	54	54	69
201.5	-1.2818	.0999581	+.1128061	-.3051605	.09571	.09576	.08235	.08838	.08840	82	88	88	73
203.5	-0.9066	.1827119	.0471056	-.5222803	.18409	.18416	.11523	.12281	.12283	115	123	123	110
205.5	-0.5315	.2977374	-.2485758	-.5002985	.50690	.50699	.14036	.14548	.14542	140	145	145	141
207.5	-0.1563	.4378990	-.3844561	-.1832776	.45228	.45241	.14870	.14819	.14812	149	148	148	139
209.5	+0.2188	.5865986	-.3708538	+.2515855	.60057	.60053	.13715	.13133	.13128	137	131	131	127
211.5	+0.5940	.7237416	-.2164208	+.5258201	.73190	.73181	.11001	.10242	.10243	110	102	102	92
213.5	0.9631	.8337514	-.0151794	.4981769	.83432	.83424	.07682	.07134	.07137	77	71	71	86
215.5	1.3443	.9105716	+.1304410	.2591660	.90566	.90561	.04666	.04486	.04491	47	45	45	60
217.5	1.7194	.9572288	.1779876	+.0068529	.95052	.95052	.02467	.02572	.02572	25	26	26	20
219.5	2.0946	.9818956	.1506825	-.1292584	.97682	.97624	.01135	.01339	.01339	11	13	13	17
221.5	2.4698	.9932405	.0963595	-.1446576	.99361	.98963	.00454	.00630	.00630	5	6	6	10
223.5	2.8449	.9977785	.0494666	-.1010456	.99591	.99593	.00158	.00266	.00265	2	3	3	4
225.5	3.2201	.9993592	.0209409	-.0530374	.99857	.99858	.00048	.00099	.00098	0	1	1	1
227.5	3.5952	.9998379	.0074250	-.0222170	.99956	.99956	.00013	.00032	.00032	0	0	0	1
229.5	3.9704	.9999642	.0022233	-.0076316	.99988	.99988				1000	998		1000

Table Showing the Successive Steps in Predicting a Distribution of ΣX When 1000 Samples ($r = 37$) Were Drawn from the Parent Distribution of *Manunculus Bulbosus* ($s = 222$).

Graph Showing Actual and Predicted Values of Σx .

Chapter III

MOMENTS OF THE DISTRIBUTION OF SECOND MOMENTS

6. Introduction

The theory of sampling as regards the mean and the standard deviation, under the assumption that the parent universe is normal, has been known for many years. It has been known from the early years of probability theory that the distribution of means of samples drawn from a normal universe follows the normal law. The moment relations of the distribution of the means of such samples were derived by Tchebycheff.

Inasmuch as we must frequently deal with samples that are taken from a universe definitely skew in character, we are at present without a definite theory to apply to such data.

In the preceding chapter we have dealt rather completely with the moments of the distribution of means of samples drawn from any universe, finite or infinite, without imposing the restrictions of normality. Our work has confirmed, as by-products, the conclusions of Tchebycheff, "Student", and Tschouproff, and has thereby greatly extended the results of their investigations.

In this chapter, in a similar manner, we wish to deal more completely with the distribution of second moments than has hitherto been accomplished. "Student" in *Biometrika*, Vol. VI, p. 3, has given formulas for the first and second moments of the distribution of second moments, and Tschouproff in *Biometrika*, Vol. XII, p. 193, has given formulas for the third and fourth moments, (I find his fourth moment in error), when sampling from an infinite universe. We shall extend the results of their investigations by an elementary method and thus make available the derivations that have previously been discouraging to the student.

7. Moments When s is Finite

Let us assume that the parent population $x_1, x_2, \dots, x_s$ is measured from their mean as origin so that

$$\sum_{i=1}^s x_i = 0.$$

Samples of r are taken. In all, we may obtain sCr samples. If $m_{2,k}$ is the second moment of the k th sample about the mean of the sample, we have

$$\begin{aligned} m_{2,k} &= \frac{\sum x^2}{r} - \frac{(\sum x)^2}{r^2} = \frac{\sum x^2}{r} - \frac{1}{r^2} [\sum x^2 + 2\sum x_i x_j] \\ &= \frac{1}{r^2} [(r-1)\sum x^2 - 2\sum x_i x_j] = \frac{1}{r^2} [(r-1)(2) - 2(1^2)] \end{aligned}$$

or finally

$$r^2 m_{2,k} = (r-1)(2) - 2(1^2)$$

when $i \neq j$ and the $\sum s$ cover only the sample.

The mean of the distribution of second moments is given by

$$\mu'_{1,2} = \frac{1}{sCr} \sum_{k=1}^{sCr} m_{2,k} = \frac{1}{r^2} [(r-1)\rho_1(2) - 2\rho_2(1^2)]$$

Employing the sigma products found in Appendix B we have

$$(16) \quad r^2 \mu'_{1,2} = s\mu_2 [(r-1)\rho_1 + \rho_2] = s^2 \rho_2 \mu_2$$

Further

$$r^4 m_{2,k}^2 = (r-1)^2 (2)^2 - 4(r-1)(2)(1^2) + 4(1^2)^2.$$

Employing the proper sigma products this becomes

$$\begin{aligned} r^4 m_{2,k}^2 &= (r-1)^2 (4) - 4(r-1)(3.1) + 2(r^2 - 2r + 3)(2^2) \\ &\quad - 4(r-3)(2.1^2) + 24(1^4), \end{aligned}$$

from which we have for the sCr sample second moments

$$r^4 \mu_{2,2}' = [(r-1)^2(4) - 4(r-1)\rho_2(3.1) + 2(r^2 - 2r + 3)\rho_2(2^2) - 4(r-3)\rho_3(2.1^2) + 24\rho_4(1^4)]$$

Now employing the moment values of the sigma functions given in Appendix B, we have

$$(17) \quad r^4 \mu_{2,2}' = s\mu_4 [(r-1)^2 \rho_1 - (r^2 - 6r + 7)\rho_2 - 4(r-3)\rho_3 - 6\rho_4] + s^2 \mu_2^2 [(r^2 - 2r + 3)\rho_2 + 2(r-3)\rho_3 + 3\rho_4]$$

Since

$$\mu_{2,2} = \mu_{2,2}' - [\mu_{1,2}']^2$$

we find

$$(18) \quad r^4 \mu_{2,2} = s\mu_4 [(r-1)^2 \rho_1 - (r^2 - 6r + 7)\rho_2 - 4(r-3)\rho_3 - 6\rho_4] + s^2 \mu_2^2 [(r^2 - 2r + 3)\rho_2 + 2(r-3)\rho_3 + 3\rho_4 - s^2 \rho_2^2]$$

from which $\sigma\mu_2$ can be found immediately.

In exactly the same manner I find $m_{2,k}^3$, which, upon using the proper sigma products of order six, becomes

$$\begin{aligned} r^6 m_{2,k}^3 &= (r-1)^3(6) - 6(r-1)^2(5.1) + 3(r-1)(r^2 - 2r + 5)(4.2) \\ &\quad - 6(r-1)(r-5)(4.1^2) - 4(3r^2 - 6r + 5)(3^2) \\ &\quad - 12(r^2 - 4r + 5)(3.2.1) + 24(3r-5)(3.1^3) \\ &\quad + 6(r^3 - 3r^2 + 9r - 15)(2^3) - 12(r^2 - 6r + 15)(2^2.1^2) \\ &\quad + 72(r-5)(2.1^4) - 720(1^6). \end{aligned}$$

Summing for all samples and dividing by sCr, we have

$$\begin{aligned} r^6 \mu_{2,k}^3 &= (r-1)^3 \rho_1(6) - 6(r-1)^2 \rho_2(5.1) + 3(r-1) \\ &\quad (r^2 - 2r + 5) \rho_2(4.2) - 6(r-1)(r-5) \rho_3(4.1^2) \end{aligned}$$

$$\begin{aligned}
& -4(3r^2-6r+5)\rho_2(3^2)-12(r^2-4r+5)\rho_3(3.2.1) \\
& +24(3r-5)\rho_4(3.1^3)+6(r^3-3r^2+9r-15)\rho_3(2^3) \\
& -12(r^2-6r+15)\rho_4(2^2.1^2)+72(r-5)\rho_5(2.1^4) \\
& -720\rho_6(1^6).
\end{aligned}$$

Now employing the moment values of the sigma functions given in Appendix B we have

$$\begin{aligned}
(19) \quad r^6\mu_{3,2}' &= s\mu_6[(r-1)^3\rho_1-(3r^3-2lr^2+45r-31)\rho_2 \\
& +2(r-3)(r^2-15r+30)\rho_3+6(3r^2-30r+65)\rho_4 \\
& +72(r-5)\rho_5+120\rho_6]+3s^2\mu_4\mu_2[(r-1) \\
& (r^2-2r+5)\rho_2-(r^3-8r^2+31r-40)\rho_3 \\
& -(5r^2-42r+95)\rho_4-18(r-5)\rho_5-30\rho_6] \\
& -2s^2\mu_3^2[(3r^2-6r+5)\rho_2-6(r^2-4r+5)\rho_3 \\
& + (3r^2-30r+65)\rho_4+12(r-5)\rho_5+20\rho_6] \\
& +s^3\mu_2^3[(r^3-3r^2+9r-15)\rho_3+3(r^2-6r+15)\rho_4 \\
& +9(r-5)\rho_5+15\rho_6]
\end{aligned}$$

By applying the relation

$$\mu_{3,2} = \mu_{3,2}' - 3\mu_{2,2}\mu_{1,2}' - (\mu_{1,2}')^3$$

we find

$$\begin{aligned}
(20) \quad r^6\mu_{3,2} &= s\mu_6[(r-1)^3\rho_1-(3r^3-2lr^2+45r-31)\rho_2 \\
& +2(r-3)(r^2-15r+30)\rho_3+6(3r^2-30r+65)\rho_4 \\
& +72(r-5)\rho_5+120\rho_6]-3s^2\mu_4\mu_2[(r-1)(r-5)\rho_2 \\
& -(r^2-6r+7)s\rho_2^2+(r^3-8r^2+31r-40)\rho_3-4(r-3)s\rho_3
\end{aligned}$$

$$\begin{aligned}
& + (5r^2 - 42r + 95)\rho_4 - 6s\rho_2\rho_4 + 18(r-5)\rho_5 + 30\rho_6] \\
& - 2s^2\mu_3^2 [(3r^2 - 6r + 5)\rho_2 - 6(r^2 - 4r + 5)\rho_3 \\
& + (3r^2 - 30r + 65)\rho_4 + 12(r-5)\rho_5 + 20\rho_6] \\
& - s^3\mu_2^3 [(r^2 - 4r + 9)s\rho_2^2 - (r^3 - 3r^2 + 9r - 15)\rho_3 \\
& - 2s^2\rho_2^3 + 6(r-3)s\rho_3 - 3(r^2 - 6r + 15)\rho_4 + 9s\rho_2\rho_4 \\
& - 9(r-5)\rho_5 - 15\rho_6].
\end{aligned}$$

While it is physically possible by the method we have employed to find the higher moments for the distribution of second moments for the case in which s is finite, it is apparent that the results obtained would be of too great length to be of value. We shall therefore leave at this point the problem of sampling from a finite population and devote our attention to the problem of sampling from an infinite universe.

8. Moments When s is Infinite

If in the results of the preceding section we let s become infinite we obtain

$$(21) \quad r^2 \mu_{1,2}^1 = r^{(2)} \mu_2 = r^{(2)} \sigma^2$$

$$(22) \quad r^4 \mu_{2,2} = r^{(2)} [(r-1)\mu_4 - (r-3)\mu_2^2] = r^{(2)} \sigma^4 [(r-1)\alpha_4 - r + 3]$$

$$\begin{aligned}
(23) \quad r^6 \mu_{2,3} = r^{(2)} \sigma^6 [(r-1)^2 \alpha_6 - 3(r-1)(r-5)\alpha_4 \\
- 2(3r^2 - 6r + 5)\alpha_3^2 + 2(r^2 - 12r + 15)]
\end{aligned}$$

We have from $m_{2,k}$ of the preceding section

$$\begin{aligned}
r^8 m_{2,3}^8 = (r-1)^4 (2)^4 - 8(r-1)^3 (2)^3 (1^2) + 24(r-1)^2 (2)^2 (1^2)^2 \\
- 32(r-1)(2)(1^2)^3 + 16(1^2)^4.
\end{aligned}$$

Summing for all samples, dividing by sCr and permitting s to become infinite we have by the Master Theorem and its corollaries

$$\begin{aligned} r^8 \mu_{4,2}^I = r^{(2)} \sigma^8 [& (r-1)^3 \alpha_8 + 4(r-1)^2 (r^2 - 2r + 7) \alpha_6 \\ & - 8(r-1) (3r^2 - 6r + 7) \alpha_5 \alpha_3 + (3r^4 - 12r^3 + 42r^2 - 60r + 35) \alpha_4^2 \\ & + 6(r-2) (r^4 - 4r^3 + 16r^2 - 40r + 35) \alpha_4 \alpha_3 - 8(r-2) \\ & (3r^3 - 15r^2 + 35r - 45) \alpha_3^2 + (r-2) (r-3) (r^4 - 4r^3 + 18r^2 \\ & - 60r + 105)] \end{aligned}$$

Since

$$\mu_{4,2} = \mu_{4,2}^I - 4\mu_{3,2} \mu_{1,2}^I - 6\mu_{2,2} (\mu_{1,2}^I)^2 - (\mu_{1,2}^I)^4$$

we obtain

$$\begin{aligned} (24) \quad r^8 \mu_{4,2} = r^{(2)} \sigma^8 [& (r-1)^3 \alpha_8 - 4(r-1)^2 (r-7) \alpha_6 - 8(r-1) \\ & (3r^2 - 6r + 7) \alpha_5 \alpha_3 + (3r^4 - 12r^3 + 42r^2 - 60r + 35) \alpha_4^2 \\ & - 6(r^4 - 7r^3 + 49r^2 - 105r + 70) \alpha_4 + 16(6r^3 - 27r^2 + 50r \\ & + 35) \alpha_3^2 + 3(r^4 - 9r^3 + 93r^2 - 255r + 210)] \end{aligned}$$

By the same laborious process I have found the values of $\mu_{5,2}$ and $\mu_{6,2}$ that are given by the following equations. These results were published by Professor Carver with due credit to the author in *Annals of Mathematical Statistics*, Vol. I, p. 271.

$$\begin{aligned} (25) \quad r^{10} \mu_{5,2} = r^{(2)} \sigma^{10} [& (r-1)^4 \alpha_{10} - 5(r-1)^3 (r-9) \alpha_8 \\ & - 40(r-1)^2 (r^2 - 2r + 3) \alpha_7 \alpha_3 + 10(r-1) (r^4 - 4r^3 \\ & + 18r^2 - 28r + 21) \alpha_6 \alpha_4 - 10(3r^5 - 27r^4 + 162r^3 - 450r^2 \\ & + 595r - 315) \alpha_4^2 - 20(r-2) (3r^4 - 24r^3 + 80r^2 - 140r \\ & + 105) \alpha_4 \alpha_3^2 + 10(5r^5 - 64r^4 + 572r^3 - 2070r^2 + 3255r \end{aligned}$$

$$\begin{aligned}
& -1890)\alpha_3^2 - 4(5r^5 - 86r^4 + 1050r^3 - 4620r^2 + 8505r \\
& - 5670) - 10(r-1)(r^4 - 7r^3 + 65r^2 - 161r + 126)\alpha_6 - 2(15r^4 \\
& - 60r^3 + 130r^2 - 140r + 63)\alpha_5^2 - 80(6r^4 - 36r^3 + 97r^2 \\
& - 126r + 63)\alpha_5\alpha_3].
\end{aligned}$$

$$\begin{aligned}
(26) \quad r^{12}\mu_{6,2} = & r^{(2)} \sigma^{12} [(r-1)^5\alpha_{12} - 6(r^5 - 15r^4 + 50r^3 - 70r^2 + 45r \\
& - 11)\alpha_{10} - 20(3r^5 - 15r^4 + 38r^3 - 54r^2 + 39r - 11)\alpha_9\alpha_3 \\
& + 15(r^6 - 6r^5 + 31r^4 - 84r^3 + 127r^2 - 102r + 33)\alpha_6\alpha_4 \\
& - 15(r^6 - 9r^5 + 96r^4 - 394r^3 + 729r^2 - 621r + 198)\alpha_8 \\
& + 2(5r^6 - 30r^5 + 165r^4 - 460r^3 + 735r^2 - 630r + 231)\alpha_6^2 \\
& - 120(r^6 - 11r^5 + 81r^4 - 294r^3 + 567r^2 - 567r + 231)\alpha_6\alpha_4 \\
& + 15(r^7 - 8r^6 + 51r^5 - 258r^4 + 815r^3 - 1540r^2 + 1645r \\
& - 770)\alpha_4^3 + 20(5r^6 - 75r^5 + 828r^4 - 3938r^3 + 9009r^2 - 9881r \\
& + 4158)\alpha_6 + 5(9r^7 - 159r^6 + 243r^5 - 26130r^4 + 135885r^3 \\
& - 35941r^2 + 474390r - 24980)\alpha_4 - 5(9r^7 - 126r^6 + 1413r^5 \\
& - 11214r^4 + 47355r^3 - 107730r^2 + 127575r - 62370)\alpha_4^2 \\
& - 24(5r^5 - 25r^4 + 70r^3 - 110r^2 + 93r - 33)\alpha_7\alpha_5 + 480(2r^5 \\
& - 15r^4 + 52r^3 - 96r^2 + 90r - 33)\alpha_7\alpha_3 - 40(3r^6 - 39r^5 + 206r^4 \\
& - 616r^3 + 1113r^2 - 1113r + 462)\alpha_6\alpha_3^2 + 24(30r^5 - 225r^4 \\
& + 820r^3 - 1610r^2 + 1638r - 693)\alpha_5^2 - 120(3r^6 - 33r^5 + 172r^4 \\
& - 530r^3 + 987r^2 - 1029r + 462)\alpha_5\alpha_4\alpha_3 + 40(51r^6 - 435r^5 \\
& + 1896r^4 - 5218r^3 + 9191r^2 - 9387r + 4158)\alpha_5\alpha_3 \\
& + 600(3r^6 - 45r^5 + 294r^4 - 1076r^3 + 2317r^2 - 2751r
\end{aligned}$$

$$\begin{aligned}
 &+1386)\alpha_4\alpha_3^2-80(21r^6-558r^5+5012r^4-22820r^3 \\
 &+57445r^2-76230r+41580)\alpha_3^2+40(9r^6-105r^5+564r^4 \\
 &-1830r^3+3745r^2-4445r+2310)\alpha_3^4-5(3r^7-59r^6 \\
 &+1136r^5-15642r^4+96135r^3-290115r^2+429030r \\
 &-249480)].
 \end{aligned}$$

9. Moments When Parent Universe is Normal

If the parent universe is normal, that is, if

$$\alpha_{2n} = \frac{(2n)!}{2^n(n!)} \quad \alpha_{2n+1} = 0$$

we have

$$(27) \left\{ \begin{aligned} \mu_{1,2} &= 0 \\ r^2\mu_{2,2} &= 2(r-1)\mu_2^2 \\ r^3\mu_{3,2} &= 8(r-1)\mu_2^3 \\ r^4\mu_{4,2} &= 12(r-1)(r-3)\mu_2^4 \\ r^5\mu_{5,2} &= 32(r-1)(5r+7)\mu_2^5 \\ r^6\mu_{6,2} &= 40(r-1)(3r^2+46r+47)\mu_2^6 \end{aligned} \right.$$

The corresponding standard moments are

$$(28) \left\{ \begin{aligned} \alpha_{1,2} &= 0 \\ \alpha_{2,2} &= 1 \\ \alpha_{3,2} &= \frac{4}{\sqrt{2(r-1)}} \\ \alpha_{4,2} &= \frac{3(r+3)}{r-1} \\ \alpha_{5,2} &= \frac{8(5r+7)}{\sqrt{2(r-1)}^3} \\ \alpha_{6,2} &= \frac{5(3r^2+46r+47)}{(r-1)^2} \end{aligned} \right.$$

These relations satisfy the condition

$$\alpha_{n+1,2} = n[\alpha_{n-1,2} + \frac{\alpha_3}{2} \alpha_{n,2}]$$

among the moments that a distribution belongs to Pearson Type III. Therefore, as far as the first six moments, the distribution of second moments of samples chosen from a normal parent universe is Pearson Type III.

The fact that the distribution of the second moments of samples chosen from an infinite normal parent universe is a Pearson Type III was first given by Helmert, F. R. in *Zeitschrift für Mathematik und Physik*, Volume XXI, (1876), page 202. It was also given by "Student", On the Probable Error of the Mean, *Biometrika*, Volume VI, (1908), page 8.

Note on $\sigma\mu_2$.

If in (22) we assume that r is so large that $\frac{1}{r^2}$ and $\frac{1}{r^3}$ may be neglected, we have

$$r\sigma^2\mu_2 = \mu_4 - \mu_2^2$$

which is the value given by Pearson, On the Probable Errors of Frequency Constants, *Biometrika*, Volume II, p. 276.

If, in addition we assume that the parent population is normal and $\mu_4 = 3\mu_2^2$, we have

$$r\sigma^2\mu_2 = 2\sigma^4$$

Chapter IV

THE DISTRIBUTION OF THIRD MOMENTS

In this chapter we shall assume our parent population to be $x_1, x_2, \dots, x_s$ so that

$$\sum_{i=1}^s x_i = 0.$$

We shall only consider the case when s becomes infinite.

Let $m_{3,k}$ denote the third moment of the k th sample about the mean of the sample. Then

$$m_{3,k} = \frac{\sum x^3}{r} - 3 \frac{\sum x^2}{r} \frac{\sum x}{r} + 2 \left(\frac{\sum x}{r} \right)^3$$

To reduce this to a form readily summable we employ

$$\sum x^2 \sum x = \sum x^3 + \sum x_i^2 x_j$$

$$(\sum x)^3 = \sum x^3 + 3 \sum x_i x_j + 6 \sum x_i x_j x_k$$

and obtain

$$r^3 m_{3,k} = (r-1)(r-2)(3) - 3(r-2)(2.1) + 12(1^3)$$

Summing for all samples, dividing by sCr and permitting s to become infinite we have by the Master Theorem and its corollaries, noting that terms with unity in the parentheses contribute zero to the results,

$$(29) \quad r^3 \mu_{1,3}' = (r-1)(r-2)r\mu_3 = r^{(3)} \mu_3 = r^{(3)} \sigma^3 \alpha_3$$

Squaring the value of $r^3 m_{3,k}$ and employing sigma products of order six found in Appendix D I find

$$\begin{aligned}
r^6 m_{3,k}^2 &= (r-1)^2 (r-2)^2 (6) - 6(r-1)(r-2)^2 (5.1) \\
&\quad - 3(r-2)^2 (2r-5)(4.2) + 6(r-2)(7r-10)(4.1^2) \\
&\quad + 2(r-2)^2 (r^2 - 2r + 10)(3^2) - 6(r-2)(r^2 - 6r + 20)(3.2.1) \\
&\quad + 24(r-2)(r-10)(3.1^3) + 18(3r^2 - 12r + 20)(2^3) \\
&\quad + 36(r^2 - 8r + 20)(2^2.1^2) - 288(r-5)(2.1^4) + 2880(1^6).
\end{aligned}$$

It should be noted that only four terms in the above equation do not contain unity in the parentheses. Hence, when the Master Theorem is employed, only these four terms contribute to our result.

Summing for all samples, dividing by sCr and permitting s to become infinite we have by the Master Theorem and its corollaries

$$\begin{aligned}
(30) \quad r^6 \mu_{2,3}^1 &= r^{(3)} \sigma^6 [(r-1)(r-2)\alpha_6 - 3(r-2)(2r-5)\alpha_4 \\
&\quad + (r-2)(r^2 - 2r + 10)\alpha_3^2 + 3(3r^2 - 12r + 20)]
\end{aligned}$$

Using

$$\mu_{2,3} = \mu_{2,3}^1 - (\mu_{1,3}^1)^2$$

we obtain

$$\begin{aligned}
(31) \quad r^6 \mu_{2,3}^2 &= r^{(3)} \sigma^6 [(r-1)(r-2)\alpha_6 - 3(r-2)(2r-5)\alpha_4 \\
&\quad - (r-2)(r-10)\alpha_3^2 + 3(3r^2 - 12r + 20)]
\end{aligned}$$

In exactly the same manner I find

$$\begin{aligned}
(32) \quad r^9 \mu_{3,3} &= r^{(3)} \sigma^9 [(r-1)^2 (r-2)^2 \alpha_9 - 9(r-1)(r-2)^2 (r-4)\alpha_7 \\
&\quad - 3(r-2)^2 (r^2 - 20r + 28)\alpha_6 \alpha_3 - 9(r-2)^2 (2r^2 - 7r + 14) \\
&\quad \alpha_5 \alpha_4 + 27(r-2)(3r^3 - 18r^2 + 56r - 56)\alpha_5 + 9(r-2)(10r^3 \\
&\quad - 121r^2 + 410r - 560)\alpha_4 \alpha_3 + 2(r^4 - 60r^3 + 472r^2 \\
&\quad - 1200r + 1120)\alpha_3^3 - 54(5r^4 - 54r^3 + 256r^2 - 600r + 560)\alpha_3]
\end{aligned}$$

$$\begin{aligned}
(33) \quad r^{12} \mu_{4,3} = & r^{(3)} o^{12} [(r-1)^3 (r-2)^3 \alpha_{12} - 6(r-1)^2 (r-2)^3 \\
& (2r-11) \alpha_{10} - 4(r-1)(r-2)^3 (r^2 - 29r + 55) \alpha_9 \alpha_3 \\
& - 9(r-2)^3 (4r^3 - 18r^2 + 60r - 55) \alpha_8 \alpha_4 + 27(r-2)^2 \\
& (6r^4 - 48r^3 + 211r^2 - 380r + 220) \alpha_8 - 36(r-2)^3 \\
& (r^3 - 6r^2 + 18r - 22) \alpha_7 \alpha_5 + 36(r-2)^2 (5r^4 - 91r^3 \\
& + 462r^2 - 1112r + 880) \alpha_7 \alpha_3 + 3(r-2)^3 (r^4 - 4r^3 \\
& + 42r^2 - 112r + 154) \alpha_6^2 - 18(r-2)^2 (2r^5 - 29r^4 \\
& + 280r^3 - 1274r^2 + 3080r - 3080) \alpha_6 \alpha_4 - 6(r-2) \\
& (r^6 - 18r^5 + 285r^4 - 2484r^3 + 9300r^2 - 17136r \\
& + 12320) \alpha_6 \alpha_3^2 + 18(r-2)(r-3)(3r^5 - 66r^4 \\
& + 602r^3 - 2828r^2 + 7000r - 6160) \alpha_6 + 108(r-2)^2 \\
& (3r^4 - 24r^3 + 119r^2 - 280r + 380) \alpha_5^2 + 36(r-2)(10r^5 \\
& - 169r^4 + 1108r^3 - 3976r^2 + 7504r - 6160) \alpha_5 \alpha_4 \alpha_3 \\
& - 108(r-2)(30r^5 - 426r^4 + 2922r^3 - 11004r^2 \\
& + 22008r - 18480) \alpha_5 \alpha_3 + 3(52r^6 - 612r^5 + 3813r^4 \\
& - 14760r^3 + 36120r^2 - 50400r + 30800) \alpha_4^3 \\
& + 27(r-3)(4r^6 - 104r^5 + 1089r^4 - 6200r^3 + 21000r^2 \\
& - 39200r + 30800) \alpha_4^2 + 9(4r^7 - 246r^6 + 5418r^5 - 4988r^4 \\
& + 247992r^3 - 710160r^2 + 1111040r - 739200) \alpha_4 \alpha_3^2 \\
& - 9(r-3)(r-4)(36r^5 - 747r^4 + 6360r^3 - 31080r^2 \\
& + 84000r - 92400) \alpha_4 + (3r^7 - 87r^6 + 2286r^5 - 31140r^4 \\
& + 190872r^3 - 612000r^2 + 1034880r - 739200) \alpha_3^4
\end{aligned}$$

$$\begin{aligned}
& -18(3r^7 - 477r^6 + 8888r^5 - 78864r^4 + 403328r^3 \\
& - 1224240r^2 + 2058560r - 1478400)\alpha_3^2 + 27(r-3) \\
& (r-4)(r-5)(9r^4 - 120r^3 + 840r^2 - 3360r + 6160)].
\end{aligned}$$

If the parent population is normal, we obtain

$$(34) \left\{ \begin{aligned} \mu_{1,3} &= 0 \\ r^4 \mu_{2,3} &= 6r^{(3)} \sigma^6 \\ \mu_{3,3} &= 0 \\ r^7 \mu_{4,3} &= 108r^{(3)} \sigma^{12} (r^2 + 27r - 70) \\ \alpha_{3,3} &= 0 \\ \alpha_{4,3} &= 3 \left[1 + \frac{6(5r-12)}{(r-1)(r-2)} \right] \end{aligned} \right.$$

This value of $\alpha_{4,3}$ approaches the value 3 as r increases. An idea of the rapidity of approach may be obtained from the following table:

r	$\alpha_{4,3}$	r	$\alpha_{4,3}$
10	12.5	250	3.4
50	4.8	500	3.2
100	3.9	1000	3.1

Since $\alpha_{3,3} = 0$ and $\alpha_{4,3} - 3$ is positive when samples of r are drawn from an infinite normal parent universe, we may fit the distribution of sample μ_3 by using the Pearson equation

$$y = y_0 \left(1 + \frac{x^2}{a^2} \right)^{-m}$$

for which

$$\begin{aligned}
\mu_n, (n \text{ odd}) &= 0, & \mu_2 &= \frac{a^2}{2m-3}, & \mu_4 &= \frac{3a^4}{(2m-5)(2m-3)} \\
\alpha_4 &= 3 + \frac{6}{2m-5}
\end{aligned}$$

Equating this value of α_4 to that given in (34) we obtain

$$m = \frac{5}{2} + \frac{(r-1)(r-2)}{6(5r-12)}$$

Now equating the value of μ_2 given in (34) to that given above we can find a^2 . Finally if N is the area under the curve we find

$$y_0 = \frac{N}{a\sqrt{\pi}} \frac{\Gamma(m)}{\Gamma(m-\frac{1}{2})}$$

Note on $\sigma\mu_3$.

If in (31) we assume r so large that $\frac{1}{r^2}$, $\frac{1}{r^3}$, etc., may be neglected we obtain

$$r\sigma^2\mu_3 = \mu_6 - \mu_3^2 - 6\mu_4\mu_2 + 9\mu_2^3$$

If, in addition, the parent population is normal we obtain

$$r\sigma^2\mu_3 = 0$$

These values agree with those of Pearson.

(See p. 26.)

Chapter V

THE DISTRIBUTION OF FOURTH MOMENTS

By a method similar to that employed in the preceding chapter I have found the first and the second moments of the distribution of fourth moments for the case in which s is infinite.

If $m_{4,k}$ denotes the fourth moment of the k th sample about the mean of the sample we have

$$m_{4,k} = \frac{\sum x^4}{r} - 4 \frac{\sum x^3}{r} \frac{\sum x}{r} + 6 \frac{\sum x^2}{r} \left(\frac{\sum x}{r} \right)^2 - 3 \left(\frac{\sum x}{r} \right)^4$$

Employing the symmetric products

$$(\sum x^2)^2 = \sum x^4 + 2\sum x_i^2 x_j^2$$

$$\sum x^2 \sum x_i x_j = \sum x_i^3 x_j + \sum x_i^2 x_j x_k$$

$$\sum x^3 \sum x = \sum x^4 + \sum x_i^3 x_j$$

we obtain

$$\begin{aligned} r^4 m_{4,k} &= (r-1)(r^2-3r+3)(4)-4(r^2-3r+3)(3.1) \\ &\quad +6(2r-3)(2^2)+12(r-3)(2.1^2)-72(1^4). \end{aligned}$$

Applying our Master Theorem, noting that only two terms contribute to our result, we have

$$(35) \quad r^4 \mu_{1,4}' = r(r-1)(r^2-35+3)\mu_4 + 3r(r-1)(2r-3)\mu_2^2 \quad \text{or}$$

$$(36) \quad r^4 \mu_{1,4} = r^{(2)} \sigma^4 [(r^2-3r+3)\alpha_4 + 3(2r-3)]$$

By squaring the value of $r^4 m_{4,k}$, and applying our Master Theorem we obtain

$$\begin{aligned}
 (37) \quad r^8 \mu_{2,4}' &= r^{(2)} \sigma^8 [(r-1)(r^2-3r+3)^2 \alpha_8 + 4(r^2-3r+3) \\
 &\quad (10r^2-27r+21) \alpha_6 - 8(r^2-3r+3)(r^3-4r^2+18r-21) \\
 &\quad \alpha_5 \alpha_3 + (r^6-8r^5+44r^4-150r^3+372r^2-540r+315) \alpha_4^2 \\
 &\quad + 6(r-2)(2r^4-27r^3+156r^2-360r+315) \alpha_4 \\
 &\quad + 8(r-2)(2r^4-24r^3+120r^2-315r+315) \alpha_3^2 \\
 &\quad + 27(r-2)(r-3)(4r^2-20r+35)]
 \end{aligned}$$

Using the known relation

$$\mu_{2,4} = \mu_{2,4}' - (\mu_{1,4}')^2$$

we obtain

$$\begin{aligned}
 (38) \quad r^8 \mu_{2,4} &= r^{(2)} \sigma^8 [(r-1)(r^2-3r+3)^2 \alpha_8 + 4(r^2-3r+3) \\
 &\quad (10r^2-27r+21) \alpha_6 - 8(r^2-3r+3)(r^3-4r^2+18r-21) \\
 &\quad \alpha_5 \alpha_3 - (r^5-23r^4+117r^3-345r^2+531r-315) \alpha_4^2 \\
 &\quad - 12(10r^4-93r^3+324r^2-513r+315) \alpha_4 + 8(r-2) \\
 &\quad (2r^4-24r^3+120r^2-315r+315) \alpha_3^2 + 18(4r^4-52r^3 \\
 &\quad + 228r^2-438r+315)]
 \end{aligned}$$

If the parent population is normal we obtain

$$(39) \quad \begin{cases} r^2 \mu_{1,4}' = 3(r-1)(r-5) \sigma^4 \\ r^5 \mu_{2,4} = 24r^{(2)} \sigma^8 (4r^2-9r+6) \end{cases}$$

It is interesting to note that if r becomes infinite, the preceding value of $\mu_{1,4}'$ approaches $3\mu_2^2$ or μ_4 .

Note on $\sigma\mu_4$.

If in (38) we assume r so large that

$\frac{1}{r^2}, \frac{1}{r^3}$, etc., may be neglected, we obtain

$$r\sigma^2\mu_4 = \mu_8 - \mu_4^2 - 8\mu_5\mu_3 + 16\mu_3^2\mu_2.$$

If, in addition, we assume the parent population normal, we obtain

$$r\sigma^2\mu_4 = 96\sigma^8$$

These values agree with those given by Pearson, On the Probable Errors of Frequency Constants, Biometrika, Volume II, page 277.

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APPENDIX

A. The Products of Sigma Functions

With the aid of the Multinomial Theorem we are able to write out such expressions as

$$(\Sigma x_i)^2 = \Sigma x_i^2 + 2\Sigma x_i x_j$$

$$(\Sigma x_i)^3 = \Sigma x_i^3 + 3\Sigma x_i^2 x_j + 6\Sigma x_i x_j x_k$$

and so on to $(\Sigma x_i)^n$.

In our problem we must evaluate sigma products to which the Multinomial Theorem will not apply. In order to facilitate the operation of multiplication of sigma functions, we have adopted the following notation:

$$\Sigma x_i = (1), \quad \Sigma x_i^2 = (2), \quad \Sigma x_i^3 = (3), \quad \dots, \quad \Sigma x_i^n = (n)$$

$$\Sigma x_i x_j = (1^2), \quad \Sigma x_i x_j x_k = (1^3), \quad \dots, \quad \Sigma x_i^m x_j^n x_k^p = (m.n.p.).$$

In this notation the following simple products are found by the Multinomial Theorem:

$$(1)^2 = (2) + 2(1^2)$$

$$(1)^3 = (3) + 3(2.1) + 6(1^3)$$

$$(1)^4 = (4) + 4(3.1) + 6(2^2) + 12(2.1^2) + 24(1^4)$$

and so on.

By ordinary multiplication the following formulas are found:

$$(1)(a) = (a+1) + (a.1), \quad a \neq 1$$

$$(1)(a.b) = (\overline{a+1}.b) + (a.\overline{b+1}) + (a.b.1), \quad a \neq b \neq 1$$

$$(1)(1^n) = (2.1^{n-1}) + \overline{n+1}(1^{n+1})$$

$$(1)(a^n) = (\overline{a+1}.a^{n-1}) + (a^n.1), \quad a \neq 1.$$

To illustrate

$$(1)(4) = (5) + (4.1)$$

$$(1)(4.2) = (5.2) + (4.3) + (4.2.1)$$

$$(1)(1^3) = (2.1^2) + 4(1^4)$$

$$(1)(2^3) = (3.2^2) + (2^3.1)$$

The following formulas also are easily derived by multiplication. If $a \neq b \neq c \neq d$,

$$(a)(b) = (a+b) + (a.b)$$

$$(a)(b.c) = (\overline{a+b}.c) + (\overline{a+c}.b) + (a.b.c)$$

$$(a)(b.c.d) = (\overline{a+b}.c.d) + (\overline{a+c}.b.d) + (\overline{a+d}.b.c) + (a.b.c.d)$$

$$(a)(b^n) = (\overline{a+b}.b^{n-1}) + (a.b^n)$$

$$(a)(b^n.c^m) = (\overline{a+b}.b^{n-1}.c^m) + (\overline{a+c}.b^n.c^{m-1}) + (a.b^n.c^m).$$

To illustrate

$$(3)(2) = (5) + (3.2)$$

$$(5)(4.3) = (9.3) + (8.4) + (5.4.3)$$

$$(3)(2^4) = (5.2^3) + (3.2^4)$$

$$(3)(4^3.2^4) = (7.4^2.2^4) + (5.4^3.2^3) + (4^3.2^4.3)$$

Noting permutations when $a = b$ we have

$$(a)(a) = (a)^2 = (2a) + 2(a^2)$$

$$(a)(a.c) = (2a.c) + (\overline{a+c}.a) + 2(a^2.c)$$

$$(a)(a^n.c) = (2a.a^{n-1}.c) + (\overline{a+c}.a^n) + \overline{n+1}(a^{n+1}.c)$$

Thus,

$$(3)(3) = (3)^2 = (6)+2(3^2)$$

$$(3)(3.2) = (6.2)+(5.3)+2(3^2.2)$$

$$(4)(4^3.3) = (8.4^2.3)+(7.4^3)+4(4^4.3)$$

Of course one may continue this algebra abstractly as far as one desires. Practice in applying these formulas will be found in the succeeding sections. We are interested here in the application of this simple theory to our problem of sampling so we shall proceed to that end.

We consider that our parent population consists of the s variates $x_1, x_2, \dots, x_s$ measured from the mean M_x as origin. That is, we have

$$\sum_{i=1}^s x_i = (1) = 0.$$

Let me emphasize that it is only when the entire parent population is under consideration that the above equation is true.

Further, in terms of moments, when the parent population is involved, we have

$$(2) = \Sigma x^2 = s\mu_2$$

$$(3) = \Sigma x^3 = s\mu_3$$

$$(3)(2) = \Sigma x^3 \Sigma x^2 = s^2 \mu_3 \mu_2$$

$$(a) = \Sigma x^a = s\mu_a$$

$$(a)(b) = s^2 \mu_a \mu_b$$

Since

$$(a_1)(a_2) = (a_1+a_2)+(a_1.a_2)$$

we have

$$\rho_2(a_1)(a_2) = \rho_2(a_1+a_2) + \rho_2(a_1.a_2)$$

But

$$(a_1) = s\mu_{a_1}, \quad (a_2) = s\mu_{a_2}, \quad (a_1+a_2) = s\mu_{a_1+a_2}$$

Therefore

$$\rho_2 s^2 \mu_{a_1} \mu_{a_2} = \rho_2 s \mu_{a_1+a_2} + \rho_2 (a_1 \cdot a_2)$$

Now let s become infinite and it follows that

$$\lim_{s \rightarrow \infty} \rho_2 (a_1 \cdot a_2) = r(2) \mu_{a_1} \mu_{a_2}$$

If $a_1 = a_2 = a$, we have the corollary

$$\lim_{s \rightarrow \infty} \rho_2 (a^2) = \frac{r(2)}{2!} \mu_a^2$$

In a similar manner we may show that

$$\lim_{s \rightarrow \infty} \rho_3 (a_1 \cdot a_2 \cdot a_3) = r(3) \mu_{a_1} \mu_{a_2} \mu_{a_3}$$

$$\lim_{s \rightarrow \infty} \rho_3 (a^3) = \frac{r(3)}{3!} \mu_a^3$$

By generalizing this procedure we have what I have termed our Master Theorem.¹

We have

$$(a_1)(a_2) \cdots (a_n) = (a_1 \cdot a_2 \cdot a_3 \cdots a_n) + R,$$

where R contains all terms of order less than n , and thus

$$\rho_n s^n \mu_{a_1} \mu_{a_2} \cdots \mu_a = \rho_n (a_1 \cdot a_2 \cdots a_n) + \rho_n R$$

Now

$$\lim_{s \rightarrow \infty} \frac{s^n}{s^{(n)}} = 1, \text{ and } \lim_{s \rightarrow \infty} \frac{R}{s^{(n)}} = 0$$

1. This theorem was given to me in 1923 by Professor H. C. Carver. It was given by A.E.R. Church in *Biometrika*, Vol. 17, 1925, p. 81.

since R can contain s to a degree not greater than (n-1).

Therefore

$$\lim_{s \rightarrow \infty} \rho_n(a_1, a_2, \dots, a_n) = r^{(n)} \mu_{a_1} \mu_{a_2} \dots \mu_{a_n}$$

If

$$a_1 = a_2 = \dots = a_k = a, \quad k < n,$$

$$\lim_{s \rightarrow \infty} \rho_n(a^k, a_{k+1}, \dots, a_n) = \frac{r^{(n)}}{k!} \mu_a^k \cdot \mu_{a_{k+1}} \cdot \dots \cdot \mu_n$$

And finally, if

$$a_1 = a_2 = \dots = a_n,$$

$$\lim_{s \rightarrow \infty} \rho_n(a^n) = \frac{r^{(n)}}{n!} \mu_a^n.$$

B. Moment Values of Sigma Products for Chapter II

In this section we shall tabulate the sigma products of various orders that we have used in our work. While we are primarily interested in the case in which (1) = 0 yet for the functions of lower orders we append the general products as well as the special values of the functions needed.

General Values

Special Values if (1)=0

Order two

$$(1)^2 = (2) + 2(1^2)$$

$$2(1^2) = -s\mu_2$$

Order three

$$(1)(2) = (3) + (2.1)$$

$$(2.1) = -s\mu_3$$

$$(1)(1^2) = (2.1) + 3(1^3)$$

$$3(1^3) = s\mu_3$$

Order four

$$(3)(1) = (4) + (3.1)$$

$$(3.1) = -s\mu_4$$

$$(2)^2 = (4) + 2(2^2)$$

$$2(2^2) = -s\mu_4 + s^2\mu_2^2$$

$$(2.1)(1) = (3.1) + 2(2^2) + 2(2.1^2) \quad 2(2.1^2) = 2s\mu_4 - s^2\mu_2^2$$

$$(1)(1^3) = (2.1^2) + 4(1^4)$$

$$8(1^4) = -2s\mu_4 + s^2\mu_2^2$$

Order five

$$(4)(1) = (5) + (4.1)$$

$$(4.1) = -s\mu_5$$

$$(3)(2) = (5) + (3.2)$$

$$(3.2) = -s\mu_5 + s^2\mu_3\mu_2$$

$$(3.1)(1) = (4.1) + (3.2) + (3.1^2) \quad 2(3.1^2) = 2s\mu_5 - s^2\mu_3\mu_2$$

$$(2^2)(1) = (3.2) + (2^2.1)$$

$$(2^2.1) = s\mu_5 - s^2\mu_3\mu_2$$

$$(2.1^2)(1) = (3.1^2) + 2(2^2.1) + 3(2.1^3) \quad 6(2.1^3) = -6s\mu_5 + 5s^2\mu_3\mu_2$$

$$(1^4)(1) = (2.1^3) + 5(1^5)$$

$$30(1^5) = 6s\mu_5 - 5s^2\mu_3\mu_2$$

For the sigma products of orders six, seven, and eight we will record the results only for the case $(1) = 0$.

Order six

$$(5.1) = -s\mu_6$$

$$(4.2) = s^2\mu_4\mu_2 - s\mu_6$$

$$2(3^2) = s^2\mu_3^2 - s\mu_6$$

$$2(4.1^2) = 2s\mu_6 - s^2\mu_4\mu_2$$

$$(3.2.1) = 2s\mu_6 - s^2\mu_4\mu_2 - s^2\mu_3^2$$

$$6(2^3) = 2s\mu_6 - 3s^2\mu_4\mu_2 + s^3\mu_2^3$$

$$6(3.1^3) = -6s\mu_6 + 3s^2\mu_4\mu_2 + 2s^2\mu_2^2$$

$$4(2^2.1^2) = -6s\mu_6 + 5s^2\mu_4\mu_2 - s^3\mu_2^3 + 2s^2\mu_3^2$$

$$24(2.1^4) = 24s\mu_6 - 18s^2\mu_4\mu_2 + 3s^3\mu_2^3 - 8s^2\mu_3^2$$

$$144(1^6) = -24s\mu_6 + 18s^2\mu_4\mu_2 - 3s^3\mu_2^3 + 8s^2\mu_3^2$$

Order seven

$$(6.1) = -s\mu_7$$

$$(5.2) = -s\mu_7 + s^2\mu_5\mu_2$$

$$(4.3) = -s\mu_7 + s^2\mu_4\mu_3$$

$$2(5.1^2) = 2s\mu_7 - s^2\mu_5\mu_2$$

$$(4.2.1) = 2s\mu_7 - s^2\mu_5\mu_2 - s^2\mu_4\mu_3$$

$$(3^2.1) = s\mu_7 - s^2\mu_4\mu_3$$

$$2(3.2^2) = 2s\mu_7 - 2s^2\mu_5\mu_2 - s^2\mu_4\mu_3 + s^3\mu_3\mu_2^2$$

$$6(4.1^3) = -6s\mu_7 + 3s^2\mu_5\mu_2 + 2s^2\mu_4\mu_3$$

$$2(3.2.1^2) = -6s\mu_7 + 3s^2\mu_5\mu_2 + 4s^2\mu_4\mu_3 - s^2\mu_3\mu_2^2$$

$$2(2^3.1) = -2s\mu_7 + 2s^2\mu_5\mu_2 + s^2\mu_4\mu_3 - s^3\mu_3\mu_2^2$$

$$24(3.1^4) = 24s\mu_7 - 12s^2\mu_5\mu_2 - 14s^2\mu_4\mu_3 + 3s^3\mu_3\mu_2^2$$

$$6(2^2.1^3) = 12s\mu_7 - 9s^2\mu_5\mu_2 - 7s^2\mu_4\mu_3 + 4s^3\mu_3\mu_2^2$$

$$120(2.1^5) = -120s\mu_7 + 84s^2\mu_5\mu_2 + 70s^2\mu_4\mu_3 - 35s^3\mu_3\mu_2^2$$

$$840(1^7) = 120s\mu_7 - 84s^2\mu_5\mu_2 - 70s^2\mu_4\mu_3 + 35s^3\mu_3\mu_2^2$$

Order eight

$$(7.1) = -s\mu_8$$

$$(6.2) = -s\mu_8 + s^2\mu_6\mu_2$$

$$(5.3) = -s\mu_8 + s^2\mu_5\mu_3$$

$$2(4^2) = -s\mu_8 + s^2\mu_4^2$$

$$2(6.1^2) = 2s\mu_8 - s^2\mu_6\mu_2$$

$$(5.2.1) = 2s\mu_8 - s^2\mu_6\mu_2 - s^2\mu_5\mu_3$$

$$(4.3.1) = 2s\mu_8 - s^2\mu_5\mu_3 - s^2\mu_4^2$$

$$2(4.2^2) = 2s\mu_8 - 2s^2\mu_6\mu_2 - s^2\mu_4^2 + s^3\mu_4\mu_2^2$$

$$2(3^2.2) = 2s\mu_8 - s^2\mu_6\mu_2 - 2s^2\mu_5\mu_3 + s^3\mu_3^2\mu_2$$

$$6(5.1^3) = -6s\mu_8 + 3s^2\mu_6\mu_2 + 2s^2\mu_5\mu_3$$

$$2(4.2.1^2) = -6s\mu_8 + 3s^2\mu_6\mu_2 + 2s^2\mu_5\mu_3 + 2s^2\mu_4^2 - s^3\mu_4\mu_2^2$$

$$4(3^2.1^2) = -6s\mu_8 + s^2\mu_6\mu_2 + 4s^2\mu_5\mu_3 + 2s^2\mu_4^2 - s^3\mu_3^2\mu_2$$

$$2(3.2^2.1) = -6s\mu_8 + 4s^2\mu_6\mu_2 + 4s^2\mu_5\mu_3 + s^2\mu_4^2 - s^3\mu_4\mu_2^2 - 2s^3\mu_3^2\mu_2$$

$$24(2^4) = -6s\mu_8 + 8s^2\mu_6\mu_2 + 3s^2\mu_4^2 - 6s^3\mu_4\mu_2^2 + s^3\mu_4^4$$

$$24(4.1^4) = 24s\mu_8 - 12s^2\mu_6\mu_2 - 8s^2\mu_5\mu_3 - 6s^2\mu_4^2 + 3s^3\mu_4\mu_2^2$$

$$6(3.2.1^3) = 24s\mu_8 - 12s^2\mu_6\mu_2 - 14s^2\mu_5\mu_3 - 6s^2\mu_4^2 + 3s^3\mu_4\mu_2^2$$

$$+ 5s^3\mu_3^2\mu_2$$

$$12(2^3.1^2) = 24s\mu_8 - 20s^2\mu_6\mu_2 - 12s^2\mu_5\mu_3 - 6s^2\mu_4^2 + 9s^3\mu_4\mu_2^2$$

$$+ 6s^3\mu_3^2\mu_2 - s^4\mu_4^4$$

$$120(3.1^5) = -120s\mu_8 + 60s^2\mu_6\mu_2 + 64s^2\mu_5\mu_3 + 30s^2\mu_4^2 - 15s^3\mu_4\mu_2^2$$

$$- 20s^3\mu_3^2\mu_2$$

$$48(2^2.1^4) = -120s\mu_8 + 84s^2\mu_6\mu_2 + 64s^2\mu_5\mu_3 + 30s^2\mu_4^2 - 33s^3\mu_4\mu_2^2$$

$$- 28s^3\mu_3^2\mu_2 + 3s^4\mu_4^4$$

$$720(2.1^6) = 720s\mu_8 - 480s^2\mu_6\mu_2 - 384s^2\mu_5\mu_3 - 130s^2\mu_4^2$$

$$+ 180s^3\mu_4\mu_2^2 + 160s^3\mu_3^2\mu_2 - 15s^4\mu_4^4$$

$$5760(1^8) = -720s\mu_8 + 480s^2\mu_6\mu_2 + 384s^2\mu_5\mu_3 + 180s^2\mu_4^2 \\ - 180s^3\mu_4\mu_2^2 - 160s^3\mu_3^2\mu_2 + 15s^4\mu_2^4$$

C. Sigma Products for Chapter III

Order four

$$(2)^2 = (4) + 2(2^2)$$

$$(2)(1^2) = (3.1) + (2.1^2)$$

$$(1^2)^2 = (2^2) + 2(2.1^2) + 6(1^4)$$

Order six

$$(2)^3 = (6) + 3(4.2) + 6(2^3)$$

$$(2)^2(1^2) = (5.1) + (4.1^2) + 2(3^2) + 2(3.2.1) + 2(2^2.1^2)$$

$$(2)(1^2)^2 = (4.2) + 3(2^3) + 2(4.1^2) + 2(3.2.1) + 4(2^2.1^2) + 6(3.1^3) \\ + 6(2.1^4)$$

$$(1^2)^3 = (3^2) + 3(3.2.1) + 15(2^2.1^2) + 6(3.1^3) + 6(2^3) + 36(2.1^4) \\ + 90(1^6)$$

Since, by the Master Theorem, the terms of a sigma product that involve unity in the parentheses contribute zero to the moment values when sampling from an infinite parent universe, we shall in the following sigma products only include the terms in a product that do not involve unity. We indicate the omitted terms involving unity by R.

Order eight

$$(2)^4 = (8) + 4(6.2) + 6(4^2) + 12(4.2^2) + 24(2^4)$$

$$(2)^3(1^2) = 3(5.3) + 6(3^2.2) + R$$

$$(2)^2(1^2)^2 = (6.2)+5(4.2^2)+2(4^2)+4(3^2.2)+12(2^4)+R$$

$$(2)(1^2)^3 = (5.3)+7(3^2.2)+6(4.2^2)+24(2^4)+R$$

$$(1^2)^4 = (4^2)+6(4.2^2)+12(3^2.2)+90(2^4)+R$$

Order ten

$$(2)^5 = (10)+5(8.2)+10(6.4)+20(6.2^2)+30(4^2.2) \\ +60(4.2^3)+120(2^5)$$

$$(2)^4(1^2) = 4(7.3)+6(5^2)+12(5.3.2)+12(4.3^2)+24(3^2.2^2)+R$$

$$(2)^3(1^2)^2 = (8.2)+3(6.4)+7(6.2^2)+6(5.3.2)+12(4^2.2) \\ +27(4.2^3)+12(4.3^2)+24(3^2.2^2)+60(2^5)+R$$

$$(2)^2(1^2)^3 = (7.3)+2(5^2)+11(5.3.2)+13(4.3^2)+56(3^2.2^2) \\ +6(6.2^2)+12(4^2.2)+42(4.2^3)+120(2^5)+R$$

$$(2)(1^2)^4 = (6.4)+13(4^2.2)+20(4.3^2)+6(6.2^2)+108(4.2^3) \\ +12(5.3.2)+120(3^2.2^2)+450(2^5)+R$$

$$(1^2)^5 = (5^2)+10(5.3.2)+20(4^2.2)+310(3^2.2^2)+30(4.3^2) \\ +180(4.2^3)+2040(2^5)+R$$

Order twelve

$$(2)^6 = (12)+6(10.2)+15(8.4)+30(8.2^2)+20(6^2) \\ +60(6.4.2)+120(6.2^3)+90(4^3)+180(4^2.2^2) \\ +360(4.2^4)+720(2^6)$$

$$(2)^5(1^2) = 5(9.3)+10(7.5)+20(7.3.2)+20(6.3^2)+30(5^2.2) \\ +30(5.4.3)+60(5.3.2^2)+60(4.3^2.2)+120(3^2.2^3)+R$$

$$\begin{aligned}
 (2)^4(1^2)^2 &= (10.2)+4(8.4)+9(8.2^2)+6(6^2)+22(6.4.2) \\
 &+48(6.2^3)+24(6.3^2)+8(7.3.2)+24(5.4.3)+12(5^2.2) \\
 &+48(5.3.2^2)+36(4^3)+78(4^2.2^2)+168(4.2^4) \\
 &+72(4.3^2.2)+144(3^4)+144(3^2.2^3)+360(2^6)+R
 \end{aligned}$$

$$\begin{aligned}
 (2)^3(1^2)^3 &= (9.3)+6(8.2^2)+3(7.5)+15(7.3.2)+19(6.3^2) \\
 &+18(6.4.2)+60(6.2^3)+24(5^2.2)+30(5.4.3) \\
 &+105(5.3.2^2)+147(4.3^2.2)+36(4^3)+108(4^2.2^2) \\
 &+288(4.2^4)+384(3^2.2^3)+144(3^4)+720(2^6)+R
 \end{aligned}$$

$$\begin{aligned}
 (2)^2(1^2)^4 &= (8.4)+6(8.2^2)+12(7.3.2)+2(6^2)+20(6.4.2) \\
 &+28(6.3^2)+126(6.2^3)+36(5.4.3)+24(5^2.2) \\
 &+192(5.3.2^2)+39(4^3)+242(4^2.2^2)+268(4.3^2.2) \\
 &+882(4.2^4)+336(3^4)+1056(3^2.2^3)+2700(2^6)+R
 \end{aligned}$$

$$\begin{aligned}
 (2)(1^2)^5 &= (7.5)+10(7.3.2)+20(6.4.2)+30(6.3^2) \\
 &+180(6.2^3)+21(5^2.2)+45(5.4.3)+360(5.3.2^2) \\
 &+60(4^3)+400(4^2.2^2)+560(4.3^2.2)+2760(4.2^4) \\
 &+3270(3^2.2^3)+720(3^4)+12240(2^6)+R
 \end{aligned}$$

$$\begin{aligned}
 (1^2)^6 &= (6^2)+15(6.4.2)+30(5^2.2)+795(4^2.2^2)+20(6.3^2) \\
 &+60(5.4.3)+90(6.2^3)+480(5.3.2^2)+90(4^3) \\
 &+1200(4.3^2.2)+1860(3^4)+11100(3^2.2^3) \\
 &+7020(4.2^4)+67950(2^6)+R
 \end{aligned}$$

D. Sigma Products for Chapter IV

Order six

$$(3)^2 = (6) + 2(3^2)$$

$$(2.1)^2 = (4.2) + 2(3^2) + 6(2^3) + R$$

$$(1^3)^2 = (2^3) + R$$

$$(3)(2.1) = (4.2) + R$$

$$(3)(1^3) = 0 + R$$

$$(2.1)(1^3) = 0 + R$$

Order nine

$$(3)^3 = (9) + 3(6.3) + 6(3^3)$$

$$(3)^2(2.1) = 2(5.4) + (7.2) + 2(4.3.2) + R$$

$$(3)^2(1^3) = 0 + R$$

$$(3)(2.1)^2 = (7.2) + (5.4) + 3(4.3.2) + 2(6.3) + 6(3^3) + 6(5.2^2) \\ + 6(3.2^3) + R$$

$$(3)(2.1)(1^3) = (4.3.2) + R$$

$$(3)(1^3)^2 = (5.2^2) + (3.2^3) + R$$

$$(2.1)^3 = (6.3) + 3(5.4) + 6(5.2^2) + 15(4.3.2) + 42(3.2^3) \\ + 12(3^3) + R$$

$$(2.1)^2(1^3) = 2(5.2^2) + 2(4.3.2) + 6(3^3) + 12(3.2^3) + R$$

$$(2.1)(1^3)^2 = (4.3.2) + 9(3.2^3) + R$$

$$(1^3)^3 = (3^3) + 6(3.2^3) + R$$

Order twelve

$$(3)^4 = (12) + 4(9.3) + 6(6^2) + 12(6.3^2) + 24(3^4)$$

$$\begin{aligned} (2.1)^4 &= (8.4) + 6(8.2^2) + 4(7.5) + 16(7.3.2) + 34(6.4.2) \\ &\quad + 48(6.3^2) + 108(6.2^3) + 6(6^2) + 48(5^2.2) + 60(5.4.3) \\ &\quad + 168(5.3.2^2) + 90(4^3) + 252(4.3^2.2) + 252(4^2.2^2) \\ &\quad + 744(4.2^4) + 216(3^4) + 648(3^2.2^3) + 2160(2^6) + R \end{aligned}$$

$$\begin{aligned} (1^3)^4 &= (4^3) + 6(4^2.2^2) + 12(4.3^2.2) + 90(4.2^4) \\ &\quad + 204(3^2.2^3) + 24(3^4) + 1860(2^6) + R \end{aligned}$$

$$\begin{aligned} (3)^2(2.1)^2 &= (10.2) + 2(9.3) + (8.4) + 6(8.2^2) + 2(7.5) \\ &\quad + 4(7.3.2) + 5(6.4.2) + 10(6.3^2) + 4(6^2) + 6(6.2^3) \\ &\quad + 6(5.4.3) + 12(5^2.2) + 12(5.3.2^2) + 10(4.3^2.2) \\ &\quad + 12(4^3) + 8(4^2.2^2) + 24(3^4) + 12(3^2.2^3) + R \end{aligned}$$

$$\begin{aligned} (3)^2(1^3)^2 &= (8.2^2) + (6.2^3) + 2(5^2.2) + 2(5.3.2^2) + 4(4^2.2^2) \\ &\quad + 2(3^2.2^3) + R \end{aligned}$$

$$\begin{aligned} (2.1)^2(1^3)^2 &= (6.4.2) + 2(6.3^2) + 15(6.2^3) + 2(5^2.2) + 2(5.4.3) \\ &\quad + 24(5.3.2^2) + 6(4^3) + 42(4^2.2^2) + 46(4.3^2.2) \\ &\quad + 208(4.2^4) + 48(3^4) + 254(3^2.2^3) + 840(2^6) + R \end{aligned}$$

$$\begin{aligned} (3)^3(2.1) &= (10.2) + 3(8.4) + 3(7.5) + 3(7.3.2) + 6(5.4.3) \\ &\quad + 6(4.3^2.2) + 3(6.4.2) + R \end{aligned}$$

$$(3)^3(1^3) = 6(4^3) + R$$

$$\begin{aligned} (3)^2(2.1)(1^3) &= (7.3.2) + 2(6.4.2) + 2(5.4.3) + 4(4.3^2.2) \\ &\quad + 4(4^2.2^2) + R \end{aligned}$$

$$\begin{aligned}
 (3)(2.1)^3 &= (9.3)+3(8.4)+6(8.2^2)+3(7.5)+15(7.3.2) \\
 &\quad +2(6^2)+14(6.3^2)+18(6.4.2)+42(6.2^3) \\
 &\quad +24(5.4.3)+12(5^2.2)+48(5.3.2^2)+48(4.3^2.2) \\
 &\quad +48(4^2.2^2)+18(4^3)+72(4.2^4)+84(3^2.2^3) \\
 &\quad +48(3^4)+R
 \end{aligned}$$

$$\begin{aligned}
 (3)(2.1)^2(1^3) &= 2(8.2^2)+2(7.3.2)+2(6.4.2)+6(6.3^2) \\
 &\quad +12(6.2^3)+3(5.4.3)+4(5^2.2)+14(5.3.2^2) \\
 &\quad +12(4^2.2^2)+6(4^3)+18(4.3^2.2)+24(4.2^4) \\
 &\quad +24(3^2.2^3)+24(3^4)+R
 \end{aligned}$$

$$\begin{aligned}
 (3)(2.1)(1^3)^2 &= (7.3.2)+(6.4.2)+9(6.2^3)+(5.4.3) \\
 &\quad +9(5.3.2^2)+6(4.3^2.2)+10(4^2.2^2) \\
 &\quad +28(4.2^4)+18(3^2.2^3)+R
 \end{aligned}$$

$$\begin{aligned}
 (3)(1^3)^3 &= (6.3^2)+6(6.2^3)+6(5.3.2^2)+3(4.3^2.2) \\
 &\quad +36(4.2^4)+4(3^4)+12(3^2.2^3)+R
 \end{aligned}$$

$$\begin{aligned}
 (2.1)(1^3)^3 &= (5.4.3)+12(5.3.2^2)+22(4.3^2.2)+168(4.2^4) \\
 &\quad +12(4^2.2^2)+36(3^4)+216(3^2.2^3)+1080(2^6)+R
 \end{aligned}$$

